

# Bose-Einstein-condensed gases in arbitrarily strong random potentials

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Bose-Einstein-condensed gases in external spatially random potentials are considered in the frame of a stochastic self-consistent mean-field approach. This method permits the treatment of the system properties for the whole range of the interaction strength, from zero to infinity, as well as for arbitrarily strong disorder. Besides a condensate and superfluid density, a glassy number density due to a spatially inhomogeneous component of the condensate occurs. For very weak interactions and sufficiently strong disorder, the superfluid fraction can become smaller than the condensate fraction, while at relatively strong interactions, the superfluid fraction is larger than the condensate fraction for any strength of disorder. The condensate and superfluid fractions, and the glassy fraction always coexist, being together either nonzero or zero. In the presence of disorder, the condensate fraction becomes a nonmonotonic function of the interaction strength, displaying an antidepletion effect caused by the competition between the stabilizing role of the atomic interaction and the destabilizing role of the disorder. With increasing disorder, the condensate and superfluid fractions jump to zero at a critical value of the disorder parameter by a first-order phase transition.

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## I. INTRODUCTION

Physics of dilute Bose gases is usually studied (see Refs. [1, 2, 3, 4, 5, 6, 7]) for asymptotically weak interactions, when the Bogolubov approximation [8, 9] is applicable. In the case of strong interactions, the most reliable techniques are purely numerical, such as Monte Carlo simulations [10, 11, 12, 13, 14, 15, 16, 17]. In the majority of experiments with trapped atoms, interactions are rather small [3, 4, 5, 6, 7] corresponding to values of the gas parameter much smaller than one. However, now it has become possible to vary the interaction strength in a wide range by using the Feshbach resonance techniques (see Refs. [4, 18]). Thus, in experiments with <sup>85</sup>Rb atoms [19, 20, 21], the value of the gas parameter 0.8 has been reached. Large values of the scattering length and, respectively, strong effective interactions can also be achieved in quasi-one-dimensional and quasi-two-dimensional configurations due to the geometric resonance [22, 23] (see discussion in Refs. [4, 7, 24, 25, 26]).

Recently, based on the idea of representative statistical ensembles [27] as applied to Bose systems with broken gauge symmetry [28, 29], a self-consistent approach has been developed [30, 31, 32, 33] for treating Bose-condensed systems with arbitrarily strong interactions. This approach was shown [33] to reproduce the weak-coupling expansions of Bogolubov [8, 9] and Lee-Huang-Yang [34, 35, 36], while simultaneously being in good agreement with numerical Monte Carlo simulations for strong interactions.

A fundamental feature of any uniform Bose system with arbitrarily strong interactions is the appearance, at low temperatures, of a Bose-Einstein condensate and, simultaneously, of superfluidity. In these systems, the condensate fraction,  $n_0$ , and the superfluid fraction,  $n_s$ , always coexist, both being nonzero below the condensation temperature. Though there is no simple relation between these fractions, the superfluid

fraction is always larger,  $n_s \geq n_0$ .

When a uniform Bose system is subject to the action of an external spatially random field, the relation between the condensate and superfluid fractions could change. Thus, Huang and Meng [37] considered a Bose-condensed system in a random external potential. They treated the case of asymptotically weak interactions and asymptotically weak disorder. Assuming that their results could be formally extended to strong disorder, they suggested that there can exist the so-called *Bose glass* phase, in the sense that there is a Bose-Einstein condensate,  $n_0 \neq 0$ , but there is no superfluidity,  $n_s \equiv 0$ . Weakly interacting Bose gas in the presence of disorder was also theoretically studied in Refs. [38, 39, 40, 41, 42, 43, 44] and the experiments demonstrating the localization of atomic matter waves were performed recently [45, 46, 47].

The arising Bose glass phase, if any, would be of high importance for the experiments with <sup>4</sup>He-filled porous media [48]. A porous material can be mimicked well by an external random potential. This is because each pore represents an external local potential. At the same time, since pores enjoy random properties, being of different sizes, shapes, and being randomly distributed in space, they do form for a Bose system a kind of a spatially random potential [48]. However, Monte Carlo simulations [49] as well as numerical calculations in the frame of the random-phase approximation [50], accomplished for a Bose system with strong disorder, though with asymptotically weak interactions, revealed no Bose glass phase. But maybe this phase could appear when both disorder and interactions were strong? In the present paper we shall give our answer to this question.

In Ref. [51], a self-consistent stochastic mean-field approximation for Bose systems in external random potentials, allowing the treatment of arbitrarily strong interactions and arbitrarily strong disorder, have been developed. It has been found that the disordered system contains several particle fractions whose relative size between 0 and 1 defines the main

system properties. There exists the fraction  $n_0$  of condensed atoms and there is the fraction  $n_N$  of normal uncondensed atoms. There also appears the fraction  $n_G$  of a glassy component. An important role is played by the anomalous average  $\sigma$ , whose absolute value  $|\sigma|$  quantifies the relative density of pair-correlated atoms [52]. Finally, there exists the superfluid fraction  $n_s$ . All these quantities are defined by the solution of a system of nonlinear equations, whose exact analysis would require numerical calculations. It is the aim of the present paper to give an analysis of the behavior of these various fractions under varying strengths of interactions and disorder in the domain where  $n_0 \neq 0$ . Throughout the paper we use the system of units, where  $\hbar \equiv 1$  and  $k_B \equiv 1$ .

## II. MAIN DEFINITIONS AND NOTATIONS

First, we must define the system to be considered. The Hamiltonian energy operator has the standard form

$$\hat{H} = \int \psi^\dagger(\mathbf{r}) \left[ -\frac{\nabla^2}{2m} + \xi(\mathbf{r}) \right] \psi(\mathbf{r}) d\mathbf{r} + \frac{\Phi_0}{2} \int \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \psi(\mathbf{r}) d\mathbf{r}, \quad (1)$$

in which  $\psi^\dagger(\mathbf{r})$  is the Bose field operator,  $\xi(\mathbf{r})$  is an external random potential, and the interaction strength is

$$\Phi_0 = 4\pi \frac{a_s}{m}, \quad (2)$$

with the scattering length  $a_s$  and the atomic mass  $m$ . The random potential is centered around zero, so that its stochastic average vanishes,

$$\langle\langle \xi(\mathbf{r}) \rangle\rangle = 0. \quad (3)$$

The stochastic correlation function has the general form

$$\langle\langle \xi(\mathbf{r}) \xi(\mathbf{r}') \rangle\rangle = R(\mathbf{r} - \mathbf{r}'). \quad (4)$$

For what follows, it is important to distinguish between the stochastic averaging over the random-field distribution, which is denoted through the angular double brackets  $\langle\langle \dots \rangle\rangle$ , as in Eqs. (3), (4), and the statistical averaging over the quantum degrees of freedom, which, for an operator  $\hat{A}$ , is denoted as

$$\langle \hat{A} \rangle_H \equiv \text{Tr} \hat{\rho} \hat{A}, \quad (5)$$

where  $\hat{\rho} = \hat{\rho}[H]$  is a statistical operator having the Gibbs form with a grand Hamiltonian  $H$ . The total averaging, given by the simple angular brackets  $\langle \dots \rangle$ , includes both the stochastic as well as the quantum averaging,

$$\langle \hat{A} \rangle \equiv \langle\langle \langle \hat{A} \rangle_H \rangle\rangle = \langle\langle \text{Tr} \hat{\rho} \hat{A} \rangle\rangle. \quad (6)$$

The appearance of a Bose-Einstein condensate implies the spontaneous gauge symmetry breaking, which can be realized by the Bogolubov shift [53, 54] of the field operator

$$\psi(\mathbf{r}) \rightarrow \hat{\psi}(\mathbf{r}) \equiv \eta(\mathbf{r}) + \psi_1(\mathbf{r}), \quad (7)$$

where  $\eta(\mathbf{r}) = \langle \hat{\psi}(\mathbf{r}) \rangle$  is the condensate wave function and  $\psi_1(\mathbf{r})$  is the Bose field operator of uncondensed atoms. The field variables  $\eta(\mathbf{r})$  and  $\psi_1(\mathbf{r})$  are treated as two independent variables, orthogonal to each other,

$$\int \eta^*(\mathbf{r}) \psi_1(\mathbf{r}) d\mathbf{r} = 0, \quad (8)$$

which excludes the double counting of the degrees of freedom. The quantum-number conservation condition

$$\langle \psi_1(\mathbf{r}) \rangle = 0 \quad (9)$$

defines  $\eta(\mathbf{r})$  as the system order parameter, equal to the average  $\langle \hat{\psi}(\mathbf{r}) \rangle$ .

These two field variables obey two normalization conditions. The condensate function is normalized to the number of condensed atoms

$$N_0 = \int |\eta(\mathbf{r})|^2 d\mathbf{r}, \quad (10)$$

while the number of uncondensed atoms

$$N_1 = \langle \hat{N}_1 \rangle \quad (11)$$

normalizes the number operator

$$\hat{N}_1 \equiv \int \psi_1^\dagger(\mathbf{r}) \psi_1(\mathbf{r}) d\mathbf{r} \quad (12)$$

for uncondensed atoms. The system stability is guaranteed by minimizing the grand thermodynamic potential

$$\Omega = -T \langle \langle \ln \text{Tr} e^{-\beta H} \rangle \rangle \quad (13)$$

under the constraints of the two normalization conditions (10) and (11), which requires the use of two Lagrange multipliers, so that the grand effective Hamiltonian in Eq. (13) is

$$H = \hat{H} - \mu_0 N_0 - \mu_1 \hat{N}_1, \quad (14)$$

with  $\hat{H}$  being the energy operator (1) under the Bogolubov shift (7). The form of the grand potential (13) corresponds to the quenched disorder.

In view of the zero-centered random potential, satisfying Eq. (3), the system can be considered as uniform on average, such that

$$\eta(\mathbf{r}) = \sqrt{\rho_0} \quad (\rho_0 \equiv N_0/V). \quad (15)$$

The field operator of uncondensed atoms can be expanded over plane waves,

$$\psi_1(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k} \neq 0} a_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}}. \quad (16)$$

The operators  $a_{\mathbf{k}}$  in the momentum representation define the normal average

$$n_{\mathbf{k}} \equiv \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle \quad (17)$$

and the anomalous average

$$\sigma_k \equiv \langle a_k a_{-k} \rangle . \quad (18)$$

The major quantities to be studied are the densities of different components. The condensate density  $\rho_0 = \rho - \rho_1$  is expressed through the total average density

$$\rho \equiv N/V = \rho_0 + \rho_1 , \quad (19)$$

where  $N$  is the total number of atoms, and through the density of uncondensed atoms

$$\rho_1 \equiv \frac{1}{V} \sum_{k \neq 0} n_k . \quad (20)$$

The anomalous average

$$\sigma_1 \equiv \frac{1}{V} \sum_{k \neq 0} \sigma_k \quad (21)$$

gives the density  $|\sigma_1|$  of pair-correlated atoms [52]. In the presence of random fields, there appears an additional important quantity, the density of the glassy component

$$\rho_G \equiv \frac{1}{V} \sum_{k \neq 0} \langle |\langle a_k \rangle_H|^2 \rangle , \quad (22)$$

whose definition is analogous to the Edwards-Andersen order parameter for spin glasses [55].

Superfluidity is characterized by the superfluid fraction

$$n_s \equiv \frac{1}{3mN} \lim_{v \rightarrow 0} \frac{\partial}{\partial \mathbf{v}} \cdot \langle \hat{\mathbf{P}}_v \rangle_v , \quad (23)$$

defined as the fraction of atoms nontrivially responding to the velocity boost with the velocity  $\mathbf{v}$ , under  $v \equiv |\mathbf{v}| \rightarrow 0$ . Here

$$\hat{\mathbf{P}}_v \equiv \hat{\mathbf{P}} + m\mathbf{v}N \quad \left( \hat{\mathbf{P}} = \sum_k \mathbf{k} a_k^\dagger a_k \right) \quad (24)$$

is the total momentum of the moving system, and the averaging in Eq. (23) implies that with the Hamiltonian of the moving system

$$H_v \equiv H + \frac{mv^2}{2} \hat{N} + \sum_k (\mathbf{k} \cdot \mathbf{v}) a_k^\dagger a_k , \quad (25)$$

where

$$\hat{N} = N_0 + \sum_{k \neq 0} a_k^\dagger a_k .$$

It can be shown (see, e.g., Refs. [3, 51]) that definition (23) yields

$$n_s = 1 - \frac{2Q}{3T} , \quad (26)$$

where

$$Q \equiv \frac{\langle \hat{\mathbf{P}}^2 \rangle}{2mN} \quad (27)$$

is the dissipated heat per atom.

### III. STOCHASTIC MEAN-FIELD APPROXIMATION

Since our aim is to treat arbitrarily strong interactions and disorder, we cannot neglect any part of the total grand Hamiltonian (14). For example, if we would omit the terms of the third and fourth order, containing the products of three and four operators  $a_k$  or  $a_k^\dagger$ , as well as the third-order term, including the product  $a_k^\dagger a_p \xi_{k-p}$ , we would come to the Bogolubov approximation, used by Huang and Meng [37], and many others, which allows the consideration of only asymptotically weak interactions and disorder. Contrary to this, we shall retain all terms of the Hamiltonian, using the stochastic mean-field approximation of Ref. [51]. This approximation was previously shown to give a very accurate description of different statistical systems with stochastic fields, as is summarized in Refs. [56, 57, 58]. The stochastic mean-field approximation for Bose systems with random fields has been described in full detail in the recent paper [51]. Therefore in the present work, we limit ourselves by mentioning only the principal points of this approximation and by reviewing the resulting formulas that are necessary for the following analysis. In the following we shall choose the condensate wave function  $\eta$  as real without restriction of generality.

The third- and fourth-order terms of the Hamiltonian with respect to the products of the operators  $a_k$  and  $a_k^\dagger$  are treated by means of the Hartree-Fock-Bogolubov approximation, similarly to the case without random fields [30, 31, 32, 33]. A special care is taken with regard to the third-order term containing the random field  $\xi_k$ . To this end, we use the fact that there are two types of averaging, the stochastic averaging, as in Eqs. (3), (4), and (50), and the quantum statistical averaging, as defined in Eq. (5). Let us introduce the random quantity

$$\alpha_k \equiv \langle a_k \rangle_H , \quad (28)$$

which is the quantum average (5) of the operator  $a_k$ . The random variable (28) is not 0, though its stochastic average

$$\langle \langle \alpha_k \rangle \rangle = \langle a_k \rangle = 0$$

is zero because of condition (9). Then, using notation (28), we accomplish a mean-field-type decoupling for the third-order term

$$a_k^\dagger a_p \xi_{k-p} = \left( \alpha_k^* a_p + a_k^\dagger \alpha_p - \alpha_k^* \alpha_p \right) \xi_{k-p} , \quad (29)$$

where only the quantum averaging is involved, but no stochastic averaging is taken. Keeping here the *stochastic* averaging unapproximated makes it possible to consider any strength of disorder.

The following important step is the use of the nonuniform and *nonlinear*, with respect to the random variable  $\xi_k$ , canonical transformation

$$a_k = u_k \hat{b}_k + v_{-k}^* \hat{b}_{-k}^\dagger - \frac{\varphi_k}{\omega_k + mc^2} , \quad (30)$$

containing a new random variable  $\varphi_k$  to be defined later. Here,

$$\begin{aligned} u_k^2 &= \frac{\omega_k + \varepsilon_k}{2\varepsilon_k}, & v_k^2 &= \frac{\omega_k - \varepsilon_k}{2\varepsilon_k}, \\ \omega_k &= \frac{k^2}{2m} + mc^2, & \varepsilon_k^2 &= \omega_k^2 - (mc^2)^2. \end{aligned} \quad (31)$$

The latter equation, for  $\varepsilon_k$ , can be represented as the Bogolubov spectrum

$$\varepsilon_k = \sqrt{(ck)^2 + \left(\frac{k^2}{2m}\right)^2}, \quad (32)$$

however with the sound velocity  $c$  differing from that of the Bogolubov form. Rather,  $c$  is given here as the solution to the equation

$$mc^2 = (\rho_0 + \sigma_1)\Phi_0. \quad (33)$$

The random variable  $\varphi_k$ , introduced in the canonical transformation (30), has to be chosen so that to simplify the total Hamiltonian. If the variable  $\varphi_k$  satisfies the Fredholm equation

$$\varphi_k = \frac{\sqrt{N_0}}{V} \xi_k - \frac{1}{V} \sum_{p \neq 0} \frac{\xi_{k-p} \varphi_p}{\omega_p + mc^2}, \quad (34)$$

then the Hamiltonian acquires the simple form

$$H = E_B + \sum_{k \neq 0} \varepsilon_k \hat{b}_k^\dagger \hat{b}_k + \sqrt{N_0} \varphi_0, \quad (35)$$

in which the quantum variables  $\hat{b}_k$  and  $\hat{b}_k^\dagger$  are separated from the random variable  $\varphi_k$ , and where the first term is a  $c$ -number quantity

$$E_B = \frac{1}{2} \sum_{k \neq 0} (\varepsilon_k - \omega_k) - \frac{N}{2} [2(\rho^2 - \rho_0^2) + (\rho_0 + \sigma_1)^2] \Phi_0.$$

The last term in Hamiltonian (35) is obtained by using Eq. (34), with the expression for  $\varphi_0$  defined by the equation

$$\varphi_0 = \frac{\sqrt{N_0}}{V} \xi_0 - \frac{1}{V} \sum_{p \neq 0} \frac{\xi_p^* \varphi_p}{\omega_p + mc^2}.$$

The relation between the random variable  $\alpha_k$ , defined in Eq. (28), and the random variable  $\varphi_k$ , satisfying Eq. (34), follows from Eq. (30), from where

$$\alpha_k = -\frac{\varphi_k}{\omega_k + mc^2}. \quad (36)$$

With Hamiltonian (35), it is straightforward to calculate the normal average (17), which gives

$$n_k = \frac{\omega_k}{2\varepsilon_k} \coth\left(\frac{\varepsilon_k}{2T}\right) - \frac{1}{2} + \langle\langle |\alpha_k|^2 \rangle\rangle, \quad (37)$$

and the anomalous average (18), yielding

$$\sigma_k = -\frac{mc^2}{2\varepsilon_k} \coth\left(\frac{\varepsilon_k}{2T}\right) + \langle\langle |\alpha_k|^2 \rangle\rangle. \quad (38)$$

The last terms in Eqs. (37) and (38) are caused by the random potential. According to relation (36), we have

$$\langle\langle |\alpha_k|^2 \rangle\rangle = \frac{\langle\langle |\varphi_k|^2 \rangle\rangle}{(\omega_k + mc^2)^2}. \quad (39)$$

The density of uncondensed atoms (20) consists of two terms,

$$\rho_1 = \rho_N + \rho_G, \quad (40)$$

the first of which is the density of normal uncondensed atoms

$$\rho_N = \frac{1}{2} \int \left[ \frac{\omega_k}{\varepsilon_k} \coth\left(\frac{\varepsilon_k}{2T}\right) - 1 \right] \frac{d\mathbf{k}}{(2\pi)^3} \quad (41)$$

and the second term is the density (22) of the glassy component

$$\rho_G = \int \frac{\langle\langle |\varphi_k|^2 \rangle\rangle}{(\omega_k + mc^2)^2} \frac{d\mathbf{k}}{(2\pi)^3}. \quad (42)$$

The anomalous average (21) also is a sum

$$\sigma_1 = \sigma_N + \rho_G \quad (43)$$

of the term

$$\sigma_N = -\frac{1}{2} \int \frac{mc^2}{\varepsilon_k} \coth\left(\frac{\varepsilon_k}{2T}\right) \frac{d\mathbf{k}}{(2\pi)^3} \quad (44)$$

and of the glassy density (42).

The superfluid fraction (26) is expressed through the dissipated heat (27). For the latter, we find

$$Q = Q_N + Q_G, \quad (45)$$

where the first term

$$Q_N = \frac{1}{8m\rho} \int \frac{k^2}{\sinh^2(\varepsilon_k/2T)} \frac{d\mathbf{k}}{(2\pi)^3} \quad (46)$$

is the heat dissipated by normal uncondensed atoms and the second term

$$Q_G = \frac{1}{2m\rho} \int \frac{k^2 \langle\langle |\varphi_k|^2 \rangle\rangle}{\varepsilon_k (\omega_k + mc^2)} \coth\left(\frac{\varepsilon_k}{2T}\right) \frac{d\mathbf{k}}{(2\pi)^3} \quad (47)$$

is the heat dissipated by the glassy component caused by the random potential.

#### IV. $\delta$ -CORRELATED RANDOM POTENTIAL

To proceed further in practical calculations, we must specify the type of random potential. For this purpose, we take the Gaussian  $\delta$ -correlated random potential with the local correlation function

$$R(\mathbf{r}) = R_0 \delta(\mathbf{r}). \quad (48)$$

Then, by means of the Fourier transformation

$$\xi(\mathbf{r}) = \frac{1}{V} \sum_{\mathbf{k}} \xi_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (49)$$

the stochastic correlator (4) reduces to

$$\langle\langle \xi_k^* \xi_p \rangle\rangle = \delta_{kp} R_0 V, \quad (50)$$

where  $V$  is the system volume. The calculation of the stochastic average  $\langle\langle |\varphi_k|^2 \rangle\rangle$ , involving the method of self-similar factor approximants [59, 60], can be done as has been thoroughly explained in Ref. [51].

For what follows, it is convenient to deal with dimensionless quantities. We introduce the notation for the condensate fraction

$$n_0 \equiv \rho_0 / \rho, \quad (51)$$

the normal fraction of uncondensed atoms

$$n_N \equiv \rho_N / \rho, \quad (52)$$

the glassy fraction

$$n_G \equiv \rho_G / \rho, \quad (53)$$

and for the dimensionless anomalous average

$$\sigma \equiv \sigma_N / \rho. \quad (54)$$

The interaction strength is characterized by the gas parameter

$$\gamma \equiv \rho^{1/3} a_s. \quad (55)$$

The dimensionless temperature is

$$t \equiv mT / \rho^{2/3}. \quad (56)$$

The strength of disorder is quantified by the disorder parameter

$$\nu \equiv \frac{7m^2 R_0}{4\pi \rho^{1/3}}. \quad (57)$$

Finally, we define the dimensionless sound velocity

$$s \equiv mc / \rho^{1/3}. \quad (58)$$

We shall consider the case  $n_0 \neq 0$ , where the gauge symmetry is spontaneously broken. The case of the unbroken gauge symmetry with  $n_0 = 0$  will be considered in future work. The sound velocity  $s$  is then well-defined and given by the solution to Eq. (33), which in the dimensionless quantities, and remembering expression (2) for the interaction strength, takes the form

$$s^2 = 4\pi\gamma(1 - n_N + \sigma). \quad (59)$$

The condensate fraction (51) can be found from Eqs. (19) and (40), which can be reduced to the equation

$$n_0 + n_N + n_G = 1. \quad (60)$$

For the normal fraction (52), taking into account Eq. (41), we have, for  $n_0 \neq 0$ ,

$$n_N = \frac{s^3}{3\pi^2} \left\{ 1 + \frac{3}{2\sqrt{2}} \int_0^\infty \left( \sqrt{1+x^2} - 1 \right)^{1/2} \times \left[ \coth \left( \frac{s^2 x}{2t} \right) - 1 \right] dx \right\}. \quad (61)$$

The anomalous average (54), with Eq. (44) and the dimensional regularization [5, 51], becomes

$$\sigma = \frac{2s^2}{\pi^{3/2}} \sqrt{\gamma n_0} - \frac{s^3}{2\sqrt{2}\pi^2} \int_0^\infty \frac{(\sqrt{1+x^2} - 1)^{1/2}}{\sqrt{1+x^2}} \times \left[ \coth \left( \frac{s^2 x}{2t} \right) - 1 \right] dx. \quad (62)$$

For the glassy fraction (53), we obtain from Eq. (42)

$$n_G = \frac{\nu(1 - n_N)}{\nu + 7s^{4/7}(s - \nu)^{3/7}}. \quad (63)$$

Finally, for the superfluid fraction (26), employing Eqs. (27), (45), (46), and (47), we find

$$n_s = 1 - \frac{4}{3} n_G - \frac{s^5}{6\sqrt{2}\pi^2 t} \int_0^\infty \frac{x(\sqrt{1+x^2} - 1)^{3/2} dx}{\sqrt{1+x^2} \sinh^2(s^2 x / 2t)}. \quad (64)$$

In the following we shall concentrate on the ground-state properties of the system, corresponding to the limit of zero temperature  $t \rightarrow 0$ . In this case, the Bose system without disorder would be completely superfluid,  $n_s = 1$ , but the condensate fraction is depleted by interactions. External random fields deplete both the condensate and superfluid fractions, so that  $n_0 < 1$  and  $n_s < 1$ . What would be the behavior of these fractions when varying the interaction and disorder strengths, that is, the gas parameter (55) and the disorder parameter (57)?

At zero temperature, the normal fraction (61) reduces to

$$n_N = \frac{s^3}{3\pi^2}, \quad (65)$$

while the anomalous fraction (62) becomes

$$\sigma = \frac{2s^2}{\pi^{3/2}} \sqrt{\gamma n_0}, \quad (66)$$

with  $n_G$  and  $n_0$  being defined from Eq. (63) and normalization (60), respectively. The superfluid fraction (64) at zero temperature takes the form

$$n_s = 1 - \frac{4}{3} n_G. \quad (67)$$

Equations (59), (60), (65), (66), and (67) determine their solutions as functions of two variables, the gas parameter (55) and the disorder parameter (57).

Let us, first, consider the asymptotic behavior of the solutions. If the disorder parameter  $\nu$  is finite and  $\gamma$  tends to

zero, there are no physical solutions, which corresponds to the stochastic instability of the ideal Bose-condensed gas [51].

For a finite disorder parameter and very strong interactions, such that  $\gamma \rightarrow \infty$ , we have

$$s \simeq s_\infty - \frac{1}{64} \left( \frac{\pi^5}{9} \right)^{1/3} \left[ 1 + \frac{\nu}{7s_\infty^{4/7} (s_\infty - \nu)^{3/7}} \right] \frac{1}{\gamma^3}, \quad (68)$$

with the limit

$$s_\infty \equiv (3\pi^2)^{1/3}. \quad (69)$$

Then the normal fraction (65) becomes

$$n_N \simeq 1 - \frac{\pi}{64} \left[ 1 + \frac{\nu}{7s_\infty^{4/7} (s_\infty - \nu)^{3/7}} \right] \frac{1}{\gamma^3}, \quad (70)$$

with the anomalous fraction (66) being

$$\sigma \simeq \frac{(9\pi)^{1/3}}{4\gamma} + O\left(\frac{1}{\gamma^4}\right). \quad (71)$$

For the glassy fraction (63), we find

$$n_G \simeq \frac{\pi\nu}{448s_\infty^{4/7} (s_\infty - \nu)^{3/7}} \left( \frac{1}{\gamma^3} \right). \quad (72)$$

Normalization (60) gives the condensate fraction

$$n_0 \simeq \frac{\pi}{64\gamma^3} + O\left(\frac{1}{\gamma^4}\right). \quad (73)$$

Equation (67) yields the superfluid fraction

$$n_s \simeq 1 - \frac{\pi\nu}{336s_\infty^{4/7} (s_\infty - \nu)^{3/7}} \left( \frac{1}{\gamma^3} \right). \quad (74)$$

As we see, strong interactions tend to destroy a Bose-Einstein condensate, suppressing  $n_0$ , but increase the superfluid fraction  $n_s$ . In this limit,  $n_s \gg n_0$ .

If the gas parameter is kept finite, but the disorder strength is getting asymptotically weak, such that  $\nu \rightarrow 0$ , then the sound velocity is

$$s \simeq s_0(1 - b\nu), \quad (75)$$

where the limit

$$s_0 \equiv (3\pi^2)^{1/3}a \quad (76)$$

depends on the interaction strength through the equation

$$a^3 + \frac{(9\pi)^{1/3}}{4\gamma} a^2 \left( 1 - \frac{8\gamma^{3/2}}{\sqrt{\pi}} \sqrt{1 - a^3} \right) = 1 \quad (77)$$

and the value of  $b$  is given by the equation

$$b \left[ (2 + a^3) \sqrt{1 - a^3} + \frac{3\sqrt{\gamma}}{\pi^{3/2}} a^3 s_0^2 \right] = \frac{\sqrt{\gamma}}{7\pi^2} (1 - a^3) s_0. \quad (78)$$

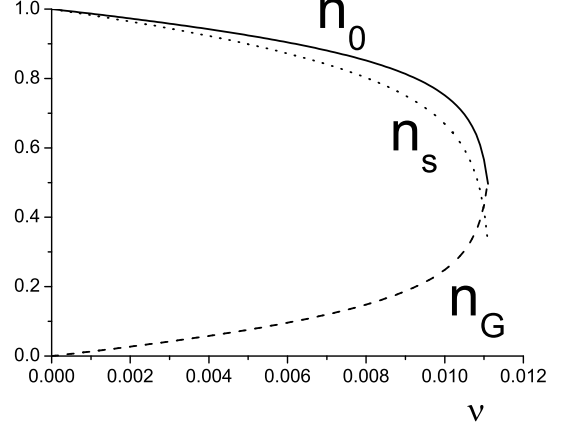


FIG. 1: Condensate fraction  $n_0$  (solid line), superfluid fraction  $n_s$  (dotted line), and glassy fraction  $n_G$  (dashed line) as functions of the dimensionless disorder parameter  $\nu$  for very weak interactions, with the gas parameter  $\gamma = 10^{-5}$ .

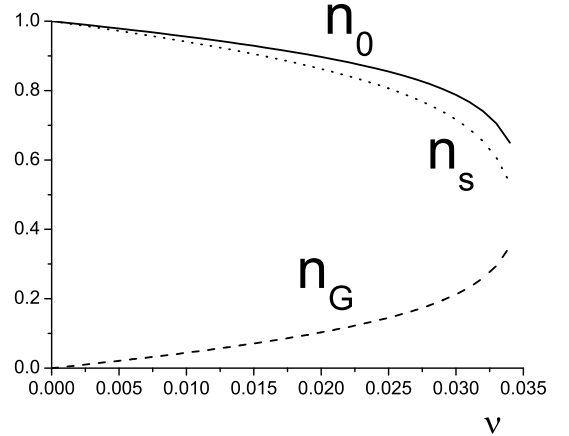


FIG. 2: The same fractions,  $n_0$ ,  $n_s$ , and  $n_G$ , as in Figure 1, but for slightly larger interactions, with the gas parameter  $\gamma = 0.001$ . Here the superfluid fraction becomes smaller than the condensate fraction, when disorder increases, but  $n_s$  is never smaller than the glassy fraction  $n_G$ .

The solutions to Eqs. (77) and (78) for weak interactions ( $\gamma \rightarrow 0$ ) are

$$\begin{aligned} a &\simeq \frac{2}{(9\pi)^{1/6}} \gamma^{1/2} + \frac{16}{(9\pi)^{2/3}} \gamma^2, \\ b &\simeq \frac{1}{7\pi^{3/2}} \gamma + O(\gamma^{5/2}) \end{aligned} \quad (79)$$

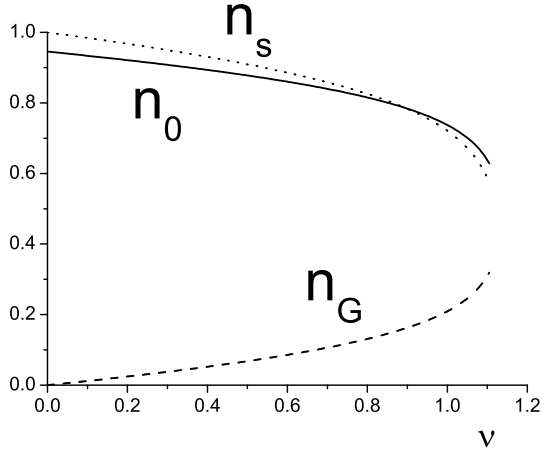


FIG. 3: Fractions  $n_0$ ,  $n_s$ , and  $n_G$  for the gas parameter  $\gamma = 0.1$ . The superfluid fraction becomes lower than the condensate fraction at the disorder parameter  $\nu \approx 0.9$ .

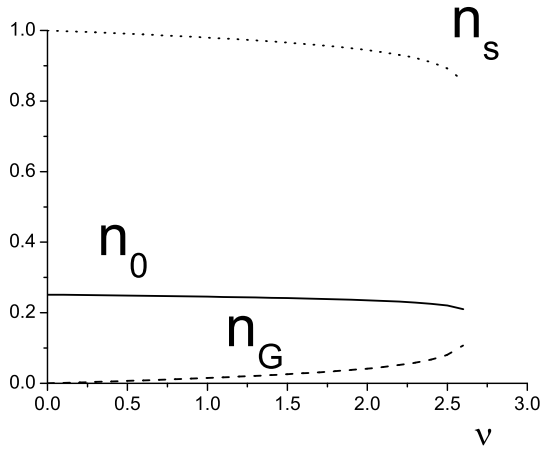


FIG. 4: Fractions  $n_0$ ,  $n_s$ , and  $n_G$  for the gas parameter  $\gamma = 0.5$ . The superfluid fraction is always higher than the condensate fraction.

and for strong interactions ( $\gamma \rightarrow \infty$ ), Eqs. (77) and (78) yield

$$a \simeq 1 - \frac{\pi}{192\gamma^3},$$

$$b \simeq \frac{1}{1344(9\pi)^{1/6}} \left( \frac{1}{\gamma^3} \right) + O\left( \frac{1}{\gamma^5} \right). \quad (80)$$

With the interaction strength varying from zero to infinity, the value of  $a$  increases from 0 to 1, so that  $0 \leq a < 1$ . But the value of  $b$  remains small for all interactions,  $b \ll 1$ . For finite  $\gamma$ , but asymptotically weak disorder, when  $\nu \rightarrow 0$ , the normal fraction is

$$n_N \simeq a^3(1 - 3b\nu), \quad (81)$$

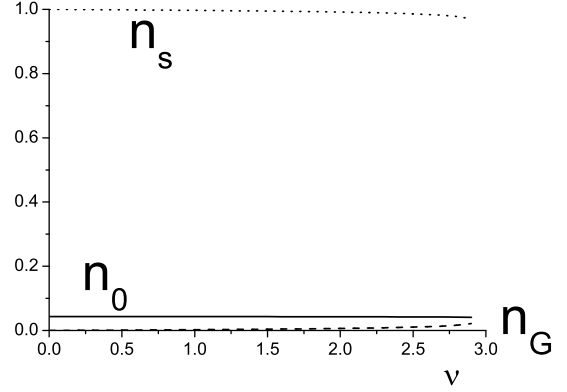


FIG. 5: Fractions  $n_0$ ,  $n_s$ , and  $n_G$  at a relatively large gas parameter  $\gamma = 1$ . The condensate and glassy fractions are strongly suppressed.

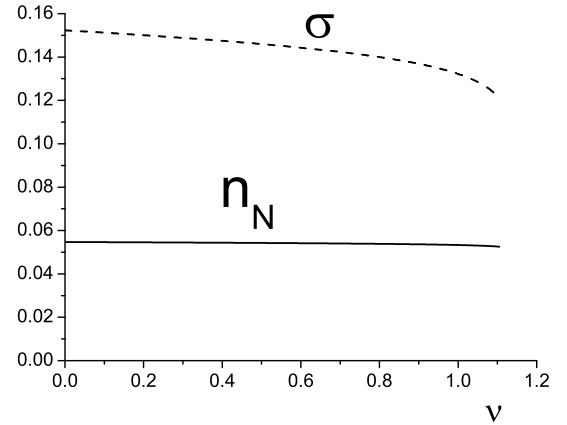


FIG. 6: Normal fraction  $n_N$  (solid line) and anomalous fraction  $\sigma$  (dashed line) as functions of the disorder parameter  $\nu$  for weak interactions, with the gas parameter  $\gamma = 0.1$ . The anomalous fraction  $\sigma$  is about three times larger than the normal fraction  $n_N$ .

while the anomalous fraction becomes

$$\sigma \simeq \frac{2s_0^2}{\pi^{3/2}} \sqrt{\gamma(1-a^3)} \left[ 1 - \frac{1-a^3 + (28-49a^3)bs_0}{14s_0} \nu \right], \quad (82)$$

where  $a$  and  $b$  are the same as above. For the glassy fraction, we have

$$n_G \simeq \frac{1-a^3}{7s_0} \nu. \quad (83)$$

The condensate fraction behaves as

$$n_0 \simeq 1 - a^3 - \frac{1-a^3 - 21a^3bs_0}{7s_0} \nu. \quad (84)$$

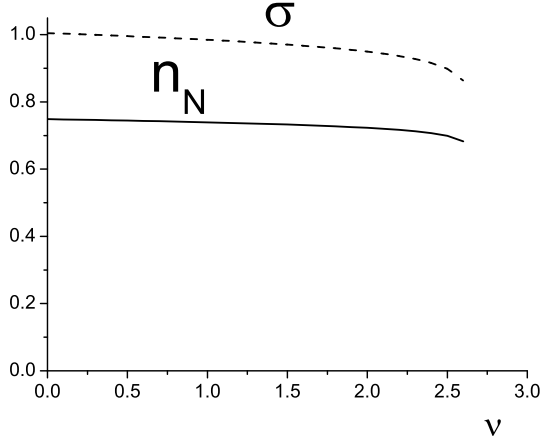


FIG. 7: Normal,  $n_N$ , and anomalous,  $\sigma$ , fractions as functions of the disorder parameter  $\nu$  for the intermediate strength, with  $\gamma = 0.5$ . The values of  $\sigma$  and  $n_N$  are close to each other, though,  $\sigma > n_N$ .

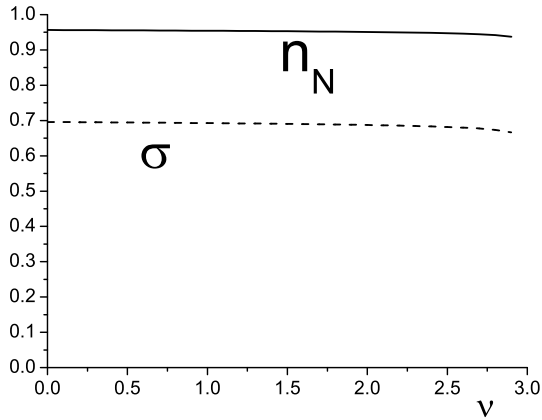


FIG. 8: Normal,  $n_N$ , and anomalous,  $\sigma$ , fractions for rather strong interactions, with the gas parameter  $\gamma = 1$ . Here, contrary to Figs. 6 and 7, the normal fraction becomes larger than the anomalous one, though they are close to each other.

For the superfluid fraction, we obtain

$$n_s \simeq 1 - \frac{4(1 - a^3)}{21s_0} \nu. \quad (85)$$

These formulas, to leading order in  $\gamma$  and  $\nu$ , coincide with the results of Ref. [37]. In the absence of disorder, the whole system would be superfluid, but never completely condensed for any finite interactions. Consequently, in the limit of  $\nu \rightarrow 0$ , we have  $n_s > n_0$ .

With increasing disorder, there occurs a first-order phase transition at a value  $\nu_c = \nu_c(\gamma)$ , when the system discontinuously transforms to a phase with unbroken gauge symmetry.

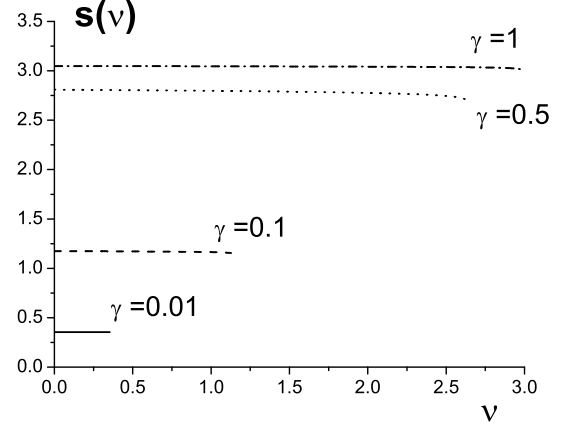


FIG. 9: Dimensionless sound velocity  $s$  as a function of the disorder parameter  $\nu$  for different gas parameters. Each line is marked by the corresponding  $\gamma$ .

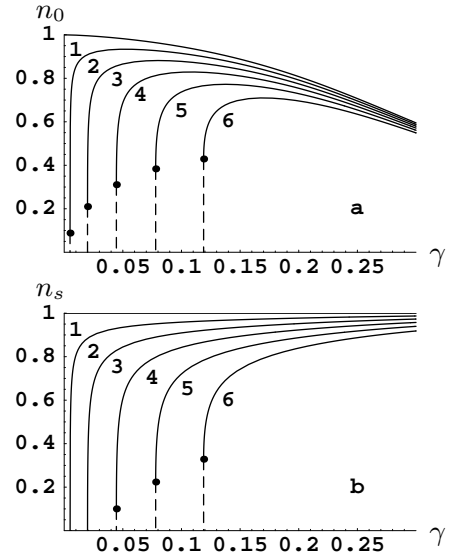


FIG. 10: Condensate fraction (a) and superfluid fraction (b) as functions of the gas parameter  $\gamma$  for  $\nu = 0$  (1), 0.25 (2), 0.5 (3), 0.75 (4), 1 (5), 1.25 (6).

At the point  $\nu_c$ , the fractions  $n_0$ ,  $n_s$ ,  $n_G$ , and  $\sigma$  jump to 0, while  $n_N$  jumps to 1. The behavior of the fractions  $n_0$ ,  $n_s$ , and  $n_G$  as functions of the disorder parameter  $\nu$  for some selected increasing values of the gas parameter  $\gamma$  are shown in Figures 1-5.

The case of very weak interactions, with  $\gamma = 10^{-5}$ , is illustrated in Fig. 1. According to Eqs. (84) and (85), we know that at asymptotically weak disorder, when  $\nu \rightarrow 0$ , the condensate fraction is smaller than the superfluid fraction,  $n_0 < n_s$ . But Fig. 1 shows that with increasing disorder the superfluid fraction becomes smaller than the condensate fraction. This occurs at a rather small value  $\nu$ , which is too close to 0 to be



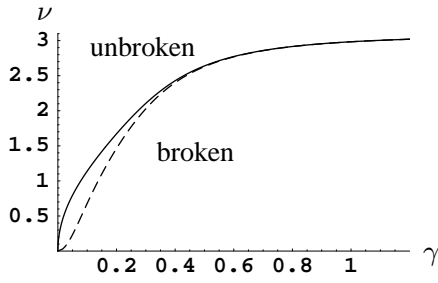


FIG. 11: Phase diagram in the  $(\gamma, \nu)$ -plane. The solid line is the line of the first-order quantum phase transition between the phases of broken and unbroken gauge symmetry. The dashed line indicates the values of  $\nu$  and  $\gamma$  at which the condensate fraction takes the maximal value (see Fig. 10a).

noticeable in the figure. Also, we see that disorder suppresses the superfluid fraction so that it becomes not merely smaller than  $n_0$ , but eventually even smaller than the glassy fraction  $n_G$ .

Increasing the interaction as we go through Figs. 2-5 strengthens superfluidity. Figure 2, for  $\gamma = 0.001$ , demonstrates that, even though there still exists a small value  $\nu$ , where the inequality  $n_0 < n_s$  changes for  $n_s < n_0$ , the superfluid fraction remains now always larger than the glassy fraction,  $n_s > n_G$ .

When increasing interactions further, say, to the gas parameter  $\gamma = 0.1$ , as in Fig. 3, the point, where the inequality  $n_s > n_0$  changes for  $n_s < n_0$ , moves to the right, getting closer to the phase transition point  $\nu_c$ . The glassy fraction  $n_G$  is always lower than both  $n_0$  and  $n_s$ .

In Fig. 4, for  $\gamma = 0.5$ , the superfluid fraction is now always larger than the condensate fraction, which distinguishes this figure from the three previous ones. The condensate fraction is yet substantially larger than the glassy fraction.

Figure 5 emphasizes how strong interactions, with  $\gamma = 1$ , favor superfluidity, while suppressing both the condensate fraction and the glassy fraction. The latter two fractions become rather small, but the superfluid fraction is close to one. This also shows that the system can be practically completely superfluid, having a tiny condensate fraction, as it happens in liquid  $^4\text{He}$ . Thus, in Fig. 5, the condensate fraction is about 5%, though the superfluid fraction is almost 100%.

The anomalous fraction  $\sigma$  and the normal fraction  $n_N$  for relatively weak interactions, with the gas parameter  $\gamma = 0.1$ , are plotted in Fig. 6. As is seen there,  $\sigma$  is about 3 times larger than  $n_N$ , which stresses the fact that  $\sigma$  cannot be neglected.

For the intermediate interaction strength, the normal and anomalous fractions are close to each other, as is shown in Fig. 7 for  $\gamma = 0.5$ . The anomalous fraction is yet larger than the normal one.

When interactions become rather strong, as in Fig. 8 for  $\gamma = 1$ , then the normal fraction surpasses the anomalous one. But, anyway,  $n_N$  is yet close to  $\sigma$ . Varying the disorder parameter does not have much influence on the values of  $n_N$  and  $\sigma$ .

The dimensionless sound velocity  $s$  as a function of the dis-

order parameter  $\nu$  for different interaction strengths is illustrated in Fig. 9. As it should be, the larger the gas parameter, the larger is the sound velocity. Stronger interactions stabilize the system, increasing the critical value  $\nu_c$  of the first-order transition. The sound velocity slightly diminishes with increasing disorder.

The condensate and superfluid fractions as functions of the interaction strength for different disorder parameters are shown in Fig. 10. As it has been emphasized earlier [51], the ideal uniform Bose-condensed gas is stochastically unstable. Finite interactions stabilize the system against weak disorder. But increasing disorder makes the system unstable, when the latter transfers through a first-order phase transition to a phase with unbroken gauge symmetry. The jumps of the condensate and superfluid fractions in Fig. 10 correspond to the phase transition. The superfluid fraction increases monotonically with the increasing interaction strength. But the remarkable fact is that, for nonvanishing disorder parameter, the condensate fraction is not a monotonic function of the interaction strength. With increasing gas parameter  $\gamma$ , the condensate fraction first increases, reaches the maximum, and then decreases. Thus, there exists the effect of *antidepletion*, when the increasing interactions result in the *rise* of the condensate fraction. This effect is due to the presence of disorder, which tends to destabilize the system, while the interaction stabilizes it. The competition between the two tendencies leads to the nonmonotonic behavior of  $n_0$ , which is seen in Fig. 10a. The line of the maxima of  $n_0$  in the  $(\gamma, \nu)$ -plane is presented as a dashed line in the phase diagram in Fig. 11.

The line  $\nu_c(\gamma)$  of the first-order phase transitions is drawn in Fig. 11. At the point  $\gamma = 0$ , corresponding to the ideal Bose gas, the phase transition is of second order. However, the ideal Bose-condensed gas is stochastically unstable, and for any infinitesimally small disorder parameter  $\nu$  it is destroyed, undergoing the phase transition to the state with  $n_0 = 0$ . Below the line  $\nu_c(\gamma)$ , there is the superfluid phase, with  $n_0 \neq 0$ ,  $n_s \neq 0$ ,  $n_G \neq 0$ ,  $\sigma \neq 0$ , and  $n_N < 1$ . Above this line, one has the phase of unbroken gauge symmetry, where  $n_0 = n_s = n_G = \sigma = 0$ , while  $n_N = 1$ . The limit of  $\nu_c(\gamma)$ , for  $\gamma$  tending to infinity, is  $(3\pi^2)^{1/3}$ . The phase transition caused by the increasing disorder is an example of a quantum phase transition.

## V. DISCUSSION

A detailed analysis of the properties of a Bose-condensed system at zero temperature in an external random potential has been presented. The disorder potential is modelled by the Gaussian uncorrelated disorder. The strength of disorder as well as the strength of interactions can be arbitrary. The system contains several fractions of particles, the condensate fraction  $n_0$ , superfluid fraction  $n_s$ , normal fraction  $n_N$ , anomalous fraction  $\sigma$ , and the fraction of a glassy component  $n_G$ . The behavior of these fractions as functions of the gas parameter and the disorder parameter was investigated. The ideal Bose-condensed gas is stochastically unstable, since any infinitesimally weak disorder destroys it, transferring it to the

normal state. Finite interactions stabilize the system. Increasing disorder leads to a first-order phase transition between the superfluid and normal phases. At asymptotically weak disorder, such that  $\nu \rightarrow 0$ , the superfluid fraction is always larger than the condensate fraction,  $n_s > n_0$ . But increasing disorder, under very weak interactions, can invert the latter inequality, when the superfluid fraction becomes lower than the condensate fraction,  $n_s < n_0$ . This is in agreement with the Monte Carlos simulations [49], where it was noticed that sufficiently strong disorder can suppress the superfluid fraction making it smaller than the condensate fraction, provided that interaction strengths are very weak. However, at sufficiently strong interactions, the superfluid fraction gets larger than the condensate fraction for all disorder parameters below the phase transition point  $\nu_c$ . To our knowledge, no numerical simulations have been accomplished, when both the disorder as well as interaction strengths would be strong.

The superfluid and condensate fractions were found always to coexist. It may occur that  $n_0 > n_s$  or  $n_0 < n_s$ , but they are nonzero or zero simultaneously. There is no pure Bose glass phase, when  $n_s$  would be zero, while  $n_0$  is nonzero, though the glassy fraction  $n_G$ , induced by disorder, is always present.

Although disorder suppresses superfluidity,  $n_s$  never becomes exactly zero, as one might conclude from the calculations for asymptotically weak disorder [37]. The pure Bose glass phase does not occur in the considered model.

The unusual effect of *antidepletion* was found, when increasing interactions can increase the condensate fraction in the presence of disorder. This effect is caused by the competition between the disorder destabilizing the system and the interactions, which stabilize the latter. As a result, in the presence of disorder, the condensate fraction becomes a nonmonotonic function of the interaction strength.

The change in the behavior of the condensed fraction  $n_0$ , normal fraction  $n_N$ , glassy fraction  $n_G$ , and the superfluid fraction  $n_s$  results from a competition between the interaction potential and the external random potential. These two causes act on the fractions in a different way. The increasing interaction always depletes the condensate, but increases the superfluid fraction. By depleting the condensate, the interaction increases the normal fraction  $n_N$ . At the same time, strengthening disorder increases the glassy fraction and depletes the condensate. The competition of all of these, sometimes contradictory, governs the overall behaviour of the fractions.

The origin of the phase transition, occurring under the increasing disorder, can be understood as follows. Recall that the ideal Bose-condensed gas is absolutely unstable with respect to any infinitesimally weak random perturbations [51]. Finite interactions do stabilize the Bose-condensed gas. But this stabilization can survive only until a finite strength of disorder, when again the system loses stability and transforms to the state where the gauge symmetry is not broken. The point is that disorder destroys coherence that is pertinent to Bose-Einstein condensation. By destroying coherence, disorder moves the system to a state with no Bose condensate.

The calculations in this paper have been done only for zero temperature. This is because, first, it has been necessary to understand the behaviour of the system under two varying pa-

rameters, the interaction strength  $\gamma$  and the strength of disorder  $\nu$ . Including temperature makes the problem dependent on three parameters. This would essentially complicate the consideration making it necessary to resort to mainly numerical calculations. We plan to present the details of these calculations in our future work. But for low temperatures, the obtained results still do hold. Including temperature just leads to more condensate depletion and enhancement of the normal fraction.

It is worth emphasizing that when analyzing the behaviour of the fractions, we always keep in mind the normalization condition (60), according to which the condensate fraction  $n_0$ , normal fraction  $n_N$ , and the glassy fraction  $n_G$  are added to 1. However, the explicit relation between the condensate and superfluid fractions is not known, because of which the latter does not enter any simple normalization condition, except that  $0 \leq n_s \leq 1$ .

In conclusion, it is important to discuss the possibility of experimental observation of the effects described in the present paper. Standard experiments are accomplished with trapped atoms. The inclusion of a trapping potential in our theory would complicate numerical investigation. However, there are two cases, when the results of our consideration could be directly applicable to experiments. First, the homogeneous picture provides a reasonable approximation for wide traps, and, second, it gives a good description of the situation at the center of a trap, even if the trap edges are rather sharp. This becomes possible because of the known fact that the local-density approximation allows for a quite accurate description of trapped atoms [1, 2, 3, 4], and the uniform case serves as a starting point for the local-density approximation.

Keeping in mind the local-density approximation, when close to the trap center the system is almost uniform, we must deal with the gas of atoms with the positive scattering length, since a homogeneous gas with attraction is known to be unstable [1, 2, 3, 4]. In experiment, one can also realize Bose-Einstein condensation of atoms with negative scattering length, provided that the atoms are trapped and their number does not exceed the critical value  $N_c$ . A simple formula for the critical number  $N_c$ , giving rather accurate estimates for harmonic traps can be represented [61] as

$$N_c = \sqrt{\frac{\pi}{2}} \frac{l_x l_y l_z}{|a_s| (l_x^2 + l_y^2 + l_z^2)},$$

where  $l_\alpha$  is the oscillator length in the  $\alpha$ -direction and the  $a_s$  scattering length. A trapped atomic cloud, with a negative scattering length, can be stable only when  $N < N_c$ . This case requires a separate investigation. In the present paper, we have considered a large system with the number of atoms not bounded from above. This is why we have assumed from the beginning that the scattering length is positive.

The value of the scattering length can be varied in a wide range, for instance, by means of the Feshbach resonance techniques [4, 18]. It would be interesting to check in experiment the behavior of the system in a fixed random potential, when the interaction strength is varied. Such a situation would correspond to Fig. 10. When diminishing the scattering length, that is, diminishing the gas parameter  $\gamma$ , we would

come to the boundary of stability of the system. Recall that, in the absence of interactions, the Bose-condensed system is stochastically unstable, such that any weak random potential destroys the condensate. This phenomenon of stochastic instability was analyzed in detail in Ref. [51]. In order to understand, why this phenomenon occurs, it is sufficient to remember that the ideal uniform Bose-condensed gas is unstable even in the absence of any random potential, which can be easily demonstrated by calculating the system compressibility and finding out that the latter diverges in the absence of interactions [4, 28].

Finally, the random potential of the type similar to that considered in the present paper can be created in experiment, e.g., by employing the optical speckle techniques[45, 46, 47]. These techniques allow for an efficient regulation of the properties of the formed random potential. It is possible to organize a frozen random distribution, independent from the time variable. It is also feasible to regulate the correlation length

characterizing the spatial properties of the speckle randomness. When the correlation length is much smaller than the healing length, the effective random potential can be represented as being  $\delta$ -correlated, which has been assumed in the present paper. At the same time, we recall that the general theory of Ref. [51] is applicable to random potentials with arbitrary correlation length, although for finite-length correlations, calculations would be essentially more complicated. In this way, it looks quite feasible to check the predictions of the suggested approach in experiments with atomic Bose gases confined in wide traps.

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